TAME AUTOMORPHISMS FIXING A VARIABLE OF FREE ASSOCIATIVE ALGEBRAS OF RANK THREE

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Devoted to the 80th anniversary of Professor Boris Plotkin.

ABSTRACT. We study automorphisms of the free associative algebra $K\langle x,y,z\rangle$ over a field K which fix the variable z. We describe the structure of the group of z-tame automorphisms and derive algorithms which recognize z-tame automorphisms and z-tame coordinates.

INTRODUCTION

Let K be an arbitrary field of any characteristic and let $K[x_1,\ldots,x_n]$ and $K\langle x_1,\ldots,x_n\rangle$ be, respectively, the polynomial algebra in n variables and of the free associative algebra of rank n, freely generated by $x_1,\ldots,x_n$. We may think of $K\langle x_1,\ldots,x_n\rangle$ as the algebra of polynomials in n noncommuting variables. The automorphism groups Aut $K[x_1,\ldots,x_n]$ and Aut $K\langle x_1,\ldots,x_n\rangle$ are well understood for $n\leq 2$ only. The description is trivial for $n=1$, when the automorphisms φ are defined by $\varphi(x_1)=\alpha x_1+\beta$, where $\alpha\in K^*=K\backslash 0$ and $\beta\in K$. The classical results of Jung–van der Kulk [J, K] for $K[x_1,x_2]$ and of Czerniakiewicz–Makar-Limanov [Cz, ML1, ML2] give that all automorphisms of $K[x_1,x_2]$ and $K\langle x_1,x_2\rangle$ are tame. Writing the automorphisms of $K[x_1,\ldots,x_n]$ and Aut $K\langle x_1,\ldots,x_n\rangle$ as n-tuples of the images of the variables, and using x,y instead of x_1,x_2, this means that Aut $K[x,y]$ and Aut $K\langle x,y\rangle$ are generated by the affine automorphisms

$$\psi=(\alpha_{11}x+\alpha_{21}y+\beta_1,\alpha_{12}x+\alpha_{22}y+\beta_2),\quad \alpha_{ij},\beta_j\in K,$$

2000 *Mathematics Subject Classification.* Primary 16S10. Secondary 16W20; 16Z05.

Key words and phrases. Automorphisms of free algebras, tame automorphisms, tame coordinates, primitive elements in free algebras.

The research of Vesselin Drensky was partially supported by Grant MI-1503/2005 of the Bulgarian National Science Fund.

The research of Jie-Tai Yu was partially supported by a Hong Kong RGC-CERG Grant.

(and $\psi_1 = (\alpha_{11}x+\alpha_{21}y, \alpha_{12}x+\alpha_{22}y)$, the linear part of ψ, is invertible) and the triangular automorphisms

$$\rho = (\alpha_1 x + p_1(y), \alpha_2 y + \beta_2), \quad \alpha_1, \alpha_2 \in K^*, p_1(y) \in K[y], \beta_2 \in K.$$

It turns out that the groups Aut $K\langle x, y\rangle$ and Aut $K[x, y]$ are naturally isomorphic. As abstract groups they are described as the free product $A *_C B$ of the group A of the affine automorphisms and the group B of triangular automorphisms amalgamating their intersection $C = A \cap B$. Every automorphism φ of $K[x, y]$ and $K\langle x, y\rangle$ can be presented as a product

$$\varphi = \psi_m^{\varepsilon_m} \rho_m \psi_{m-1} \cdots \rho_2 \psi_1 \rho_1^{\varepsilon_1}, \tag{1}$$

where $\psi_i \in A$, $\rho_i \in B$ (ε_1 and ε_m are equal to 0 or 1), and, if φ does not belong to the union of A and B, we may assume that $\psi_i \in A \backslash B$, $\rho_i \in B \backslash A$. The freedom of the product means that if φ has a nontrivial presentation of this form, then it is different from the identity automorphism.

In the case of arbitrary n, the tame automorphisms are defined similarly, as compositions of affine and triangular automorphisms. One studies not only the automorphisms but also the coordinates, i.e., the automorphic images of x_1.

We shall mention few facts related with the topic of the present paper, for z-automorphisms of $K[x, y, z]$ and $K\langle x, y, z\rangle$, i.e., automorphisms fixing the variable z. For more details we refer to the books by van den Essen [E], Mikhalev, Shpilrain, and Yu [MSY], and our survey article [DY1].

Nagata [N] constructed the automorphism of $K[x, y, z]$

$$\nu = (x - 2(y^2 + xz)y - (y^2 + xz)^2 z, y + (y^2 + xz)z, z)$$

which fixes z. He showed that ν is nontame, or wild, considered as an automorphism of $K[z][x, y]$, and conjectured that it is wild also as an element of Aut $K[x, y, z]$. This was the beginning of the study of z-automorphisms.

It is relatively easy to see (and to decide algorithmically) whether an endomorphism of $K[z][x, y]$ is an automorphism and whether this automorphism is z-tame, or tame as an automorphism of $K[z][x, y]$. When char $K = 0$, Drensky and Yu [DY2] presented a simple algorithm which decides whether a polynomial $f(x, y, z) \in K[x, y, z]$ is a z-coordinate and whether this coordinate is z-tame. This provided many new wild automorphisms and wild coordinates of $K[z][x, y]$. These results in

[DY2] are based on a similar algorithm of Shpilrain and Yu [SY1] which recognizes the coordinates of $K[x,y]$. Shestakov and Umirbaev [SU1, SU2, SU3] established that the Nagata automorphism is wild. They also showed that every wild automorphism of $K[z][x,y]$ is wild as an automorphism of $K[x,y,z]$. Umirbaev and Yu [UY] proved that the z-wild coordinates in $K[z][x,y]$ are wild also in $K[x,y,z]$. In this way, all z-wild examples in [DY2] give automatically wild examples in $K[x,y,z]$.

Going to free algebras, the most popular candidate for a wild automorphism of $K\langle x,y,z\rangle$ is the example of Anick $(x+(y(xy-yz),y,z+(zy-yz)y)\in \text{Aut } K\langle x,y,z\rangle$, see the book by Cohn [C], p. 343. It fixes one variable and its abelianization is a tame automorphism of $K[x,y,z]$. Exchanging the places of y and z, we obtain the automorphism $(x+z(xz-zy),y+(xz-zy)z,z)$ which fixes z (or a z-automorphism), and refer to it as the Anick automorphism. It is linear in x and y, considering z as a "noncommutative constant". Drensky and Yu [DY3] showed that such z-automorphisms are z-wild if and only if a suitable invertible 2×2 matrix with entries from $K[z_1,z_2]$ is not a product of elementary matrices. In particular, this gives that the Anick automorphism is z-wild. When char $K=0$, Umirbaev [U] described the defining relations of the group of tame automorphisms of $K[x,y,z]$. He showed that $\varphi=(f,g,h)\in \text{Aut } K\langle x,y,z\rangle$ is wild if the endomorphism $\varphi_0=(f_0,g_0,z)$ of $K\langle x,y,z\rangle$ is a z-wild automorphism, where f_0,g_0 are the linear in x,y components of f,g, respectively. This implies that the Anick automorphism is wild. Recently Drensky and Yu [DY4, DY5] established the wildness of a big class of automorphisms and coordinates of $K\langle x,y,z\rangle$. Many of them cannot be handled with direct application of the methods of [DY3] and [U]. These results motivate the needs of systematic study of z-automorphisms of $K\langle x,y,z\rangle$. As in the case of z-automorphisms of $K[x,y,z]$, they are simpler than the arbitrary automorphisms of $K\langle x,y,z\rangle$ and provide important examples and conjectures for Aut $K\langle x,y,z\rangle$.

In the present paper we describe the structure of the group of z-tame automorphisms of $K\langle x,y,z\rangle$ as the free product of the groups of z-affine automorphisms and z-triangular automorphisms amalgamating the intersection. We also give algorithms which recognize z-tame automorphisms and coordinates of $K\langle x,y,z\rangle$. As an application, we show that all the z-automorphisms of the form $\sigma_h=(x+zh(xz-zy,z),y+h(xz-zy,z)z)$ are z-wild when the polynomials $h(xz-zy,z)$ are of positive

degree in x. This kind of automorphisms appear in [DY4, DY5] but the considerations there do not cover the case when $h(xz-zy,z)$ belongs to the square of the commutator ideal of $K\langle x,y,z\rangle$. Besides, the polynomial $x+zh(xz-zy,z)$ is a z-wild coordinate. Finally, we show that the z-endomorphisms of the form $\varphi=(x+u(x,y,z),y+v(x,y,z))$, where $(u,v)\neq(0,0)$ and all monomials of u and v depend on both x and y, are not automorphisms. A partial case of this result was an essential step in the proof of the theorem of Czerniakiewicz and Makar-Limanov for the tameness of Aut $K\langle x,y\rangle$. The paper may be considered as a continuation of our paper [DY3].

1. The group of z-tame automorphisms

We fix the field K and consider the free associative algebra $K\langle x,y,z\rangle$ in three variables. We call the automorphism φ of $K\langle x,y,z\rangle$ a z-automorphism if $\varphi(z)=z$, and denote the automorphism group of the z-automorphisms by $\mathrm{Aut}_z\langle x,y,z\rangle$. Since we want to emphasize that we work with z-automorphisms, we shall write $\varphi=(f,g)$, omitting the third coordinate z. The multiplication will be from right to left. If $\varphi,\psi\in\mathrm{Aut}_zK\langle x,y,z\rangle$, then in $\varphi\psi$ we first apply ψ and then φ. Hence, if $\varphi=(f,g)$ and $\psi=(u,v)$, then

$$\varphi\psi=(u(f,g,z),v(f,g,z)).$$

The z-affine and z-triangular automorphisms of $K\langle x,y,z\rangle$ are, respectively, of the form

$$\psi=(\alpha_{11}x+\alpha_{21}y+\alpha_{31}z+\beta_1,\alpha_{12}x+\alpha_{22}y+\alpha_{32}z+\beta_2),$$

$\alpha_{ij},\beta_j\in K$, the 2×2 matrix $(\alpha_{ij})_{i,j=1,2}$ being invertible,

$$\rho=(\alpha_1x+p_1(y,z),\alpha_2y+p_2(z)),$$

$\alpha_j\in K^*$, $p_1\in K\langle y,z\rangle$, $p_2\in K[z]$. The affine and the triangular z-automorphisms generate, respectively, the subgroups A_z and B_z of $\mathrm{Aut}_zK\langle x,y,z\rangle$. We denote by $\mathrm{TAut}_zK\langle x,y,z\rangle$ the group of z-tame automorphisms which is generated by the z-affine and z-triangular automorphisms. Of course, we may define the z-affine automorphisms as the z-automorphisms of the form $\psi=(f,g)$, where the polynomials $f,g\in K\langle x,y,z\rangle$ are linear in x and y. But, as we commented in [DY3], this definition is not convenient. For example, the Anick automorphism is affine in this sense but is wild.

In the commutative case, the z-automorphisms of $K[x,y,z]$ are simply the automorphisms of the $K[z]$-algebra $K[z][x,y]$. A result of

Wright [Wr] states that over any field K the group $\mathrm{TAut}_z K[x,y,z]$ has the amalgamated free product structure

$$\mathrm{TAut}_z K[x,y,z] = A_z *_{C_z} B_z,$$

where A_z and B_z are defined as in the case of $K\langle x,y,z\rangle$ and $C_z = A_z \cap B_z$. (The original statement in [Wr] holds in a more general situation. In the case of $K[x,y,z]$ it involves affine and linear automorphisms with coefficients from $K[z]$ but this is not essential because every invertible matrix with entries in $K[z]$ is a product of elementary and diagonal matrices.)

Every z-tame automorphism φ of $K\langle x,y,z\rangle$ can be presented as a product in the form (1) where $\psi_i \in A_z$, $\rho_i \in B_z$ (ε_1 and ε_m are equal to 0 or 1), and, if φ does not belong to the union of A_z and B_z, we may assume that $\psi_i \in A_z \backslash B_z$, $\rho_i \in B_z \backslash A_z$. Fixing the linear nontriangular z-automorphism $\tau = (y,x)$, we can present φ in the canonical form

$$\varphi = \rho_n \tau \cdots \tau \rho_1 \tau \rho_0, \tag{2}$$

where $\rho_0, \rho_1, \ldots, \rho_n \in B_z$ and only ρ_0 and ρ_n are allowed to belong to A_z, see for example p. 350 in [C]. Let

$$\rho_i = (\alpha_i x + p_i(y,z), \beta_i y + r_i(z)), \quad \alpha_i, \beta_i \in K^*, p_i \in K\langle y,z\rangle, r_i \in K[z].$$

Using the equalities for compositions of automorphisms

$$(\alpha x + p(y,z), \beta y + r(z)) = (x + \alpha^{-1}(p(y,z) - p(0,z)), y)(\alpha x + p(0,z), \beta y + r(z)),$$

$$(\alpha x + p(z), \beta y + r(z))\tau = (\beta y + r(z), \alpha x + p(z)) = \tau(\beta x + r(z), \alpha y + p(z)),$$

$p(z), r(z) \in K[z]$, we can do further simplifications in (2), assuming that $\rho_1, \ldots, \rho_{n-1}$ are not affine and, together with ρ_n, are of the form $\rho_i = (x + p_i(y,z), y)$ with $p_i(0,z) = 0$ for all $i = 1, \ldots, n$. We also assume that $\rho_0 = (\alpha_0 x + p_0(y,z), \beta_0 y + r_0(z))$. The condition that $\rho_1, \ldots, \rho_{n-1}$ are not affine means that $\deg_y p_i(y,z) \geq 1$ and if $\deg_y p_i(y,z) = 1$, then $\deg_z p_i(y,z) \geq 1$, $i = 1, \ldots, n-1$.

The following result shows that the structure of the group of z-tame automorphisms of $K\langle x,y,z\rangle$ is similar to the structure of the group of z-tame automorphsims fo $K[x,y,z]$.

Theorem 1.1. *Over an arbitrary field K, the group $\mathrm{TAut}_z K\langle x,y,z\rangle$ of z-tame automorphisms of $K\langle x,y,z\rangle$ is isomorphic to the free product $A_z *_{C_z} B_z$ of the group A_z of the z-affine automorphisms and the group B_z of z-triangular automorphisms amalgamating their intersection $C_z = A_z \cap B_z$.*

Proof. We define a bidegree of $K\langle x,y,z\rangle$ assuming that the monomial w is of bidegree $\text{bideg}\, w=(d,e)$ if $\deg_x w+\deg_y w=d$ and $\deg_z w=e$. We order the bidegrees (d,e) lexicographically, i.e., $(d_1,e_1)>(d_2,e_2)$ means that either $d_1>d_2$ or $d_1=d_2$ and $e_1>e_2$. We denote by $\overline{p}$ the leading bihomogeneous component of the nonzero polynomial $p(x,y,z)$. Let $\varphi=(f,g)$ be in the form (2), with all the restrictions fixed above, and let $q_i(y,z)$ be the leading component of $p_i(y,z)$. Direct computations give that, if ρ_n is not linear and $p_0(y,z)\neq\gamma_0y+p_0'(z)$ in $\rho_0=(\alpha_0x+p_0(y,z),\beta_0y+r_0(z))$, then

$$\overline{f}=q_0(q_1(\dots q_{n-1}(q_n(y,z),z)\dots,z),z),$$
$$\overline{g}=\beta_0q_1(\dots q_{n-1}(q_n(y,z),z)\dots,z), \tag{3}$$

and $\text{bideg}\,\overline{f}>(1,0)$. Hence φ is not the identity automorphism. Similar considerations work when at least one of the automorphisms ρ_0 and ρ_n is affine. For example, if $\rho_0=(\alpha_0+\gamma_0y+p_0'(z),\beta_0y+r_0(z))$, $\gamma_0\in K^*$, and $\text{bideg}\,p_n(y,z)>(1,0)$, then

$$\overline{f}=\gamma_0q_1(\dots q_{n-1}(q_n(y,z),z)\dots,z),$$
$$\overline{g}=\beta_0q_1(\dots q_{n-1}(q_n(y,z),z)\dots,z).$$

If $\text{bideg}\,p_0(y,z)>(1,0)$ and $\rho_n=(x+\gamma_ny,y)$, $\gamma_n\in K^*$, then

$$\overline{f}=q_0(q_1(\dots q_{n-1}(q_n(x+\gamma y,z),z)\dots,z),z),$$
$$\overline{g}=\beta_0q_1(\dots q_{n-1}(q_n(x+\gamma y,z),z)\dots,z).$$

In all the cases, φ is not the identity automorphism. Hence, if φ has a nontrivial presentation in the form (2), then it is different from the identity automorphism, and we conclude that $\text{TAut}_zK\langle x,y,z\rangle$ is a free product with amalgamation of the groups A_z and B_z. $\square$

Following our paper [DY3] we identify the group of z-automorphisms which are linear in x and y with the group $GL_2(K[z_1,z_2])$. Let $f\in K\langle x,y,z\rangle$ be linear in x,y. Then f has the form

$$f=\sum\alpha_{ij}z^ixz^j+\sum\beta_{ij}z^iyz^j,\quad \alpha_{ij},\beta_{ij}\in K.$$

The z-derivatives f_x and f_y are defined by

$$f_x=\sum\alpha_{ij}z_1^iz_2^j,\quad f_y=\sum\beta_{ij}z_1^iz_2^j.$$

Here f_x and f_y are in $K[z_1,z_2]$ and are polynomials in two commuting variables. The z-Jacobian matrix of the linear z-endomorphism $\varphi=$

(f, g) of $K\langle x, y, z\rangle$ is defined as

$$J_z(\varphi) = \begin{pmatrix} f_x & g_x \\ f_y & g_y \end{pmatrix}.$$

By [DY3] the mapping $\varphi \to J_z(\varphi)$ is an isomorphism of the group of the z-automorphisms which are linear in x, y and $GL_2(K[z_1, z_2])$. Also, such an automorphism is z-tame if and only if its z-Jacobian matrix belongs to $GE_2(K[z_1, z_2])$. (By the further development of this result by Umirbaev [U], the z-wild automorphisms of the considered type are wild also as automorphisms of $K\langle x, y, z\rangle$.)

Corollary 1.2. *The group* $\mathrm{TAut}_z K\langle x, y, z\rangle$ *is isomorphic to the free product with amalgamation* $GE_2(K[z_1, z_2]) *_{C_1} B_z$*, where* $GE_2(K[z_1, z_2])$ *is identified as above with the group of* z*-tame automorphisms which are linear in* x *and* y*, and* $C_1 = GE_2(K[z_1, z_2]) \cap B_z$*.*

Proof. Everything follows from the observations that: (i) in the form (2), $\rho_j\tau\cdots\tau\rho_i \in GE_2(K[z_1, z_2])$ if and only if all $\rho_j, \ldots, \rho_i$ belong to $GE_2(K[z_1, z_2])$; (ii) $\rho_j\tau\cdots\tau\rho_i \in C_1$ if and only if $i = j$ and $\rho_i \in GE_2[z_1, z_2]$; (iii) $\tau \in GE_2(K[z_1, z_2])$. □

2. Recognizing z-tame automorphisms and coordinates

Now we use Theorem 1.1 to present algorithms which recognize z-tame automorphisms and coordinates of $K\langle x, y, z\rangle$. Of course, in all algorithms we assume that the field K is constructive. We start with an algorithm which determines whether a z-endomorphism of $K\langle x, y, z\rangle$ is a z-tame automorphism. The main idea is similar to that of the well known algorithm which decides whether an endomorphism of $K[x, y]$ is an automorphism, see Theorem 6.8.5 in [C], but the realization is more sophisticated. In order to simplify the considerations, we shall use the trick introduced by Formanek [F] in his construction of central polynomials of matrices.

Let H_n be the subspace of $K\langle x, y, z\rangle$ consisting of all polynomials which are homogeneous of degree n with respect to x and y. We define an action of $K[t_0, t_1, \ldots, t_n]$ on H_n in the following way. If

$$w = z^{a_0}u_1z^{a_1}u_2\cdots z^{a_{n-1}}u_nz^{a_n},$$

where $u_i = x$ or $u_i = y$, $i = 1, \ldots, n$, then

$$t_0^{b_0}t_1^{b_1}\cdots t_n^{b_n} * w = z^{a_0+b_0}u_1z^{a_1+b_1}u_2\cdots z^{a_{n-1}+b_{n-1}}u_nz^{a_n+b_n},$$

and then extend this action by linearity. Clearly, H_n is a free $K[t_0, t_1, \ldots, t_n]$-module with basis consisting of the 2^n monomials $u_1 \cdots u_n$, where $u_i = x$ or $u_i = y$. The proof of the following lemma is obtained by easy direct computation.

Lemma 2.1. *Let $\beta \in K^*$,*

$$v(x,y,z) = \sum \theta_i(t_0, t_1, \ldots, t_k) * u_{i_1} \cdots u_{i_k} \in H_k, \tag{4}$$

$$q(y,z) = \omega(t_0, t_1, \ldots, t_d) * y^d \in H_d, \tag{5}$$

where $\theta_i \in K[t_0, t_1, \ldots, t_k]$, $\omega \in K[t_0, t_1, \ldots, t_d]$, $u_{i_j} = x$ or $u_{i_j} = y$. Then

$$u(x,y,z) = q(v(x,y,z)/\beta, z) = \omega(t_0, t_d, t_{2d}, \ldots, t_{kd})/\beta^d$$

$$\begin{gathered} \left(\sum \theta_i(t_0, t_1, \ldots, t_k) * u_{i_1} \cdots u_{i_k}\right) \\ \left(\sum \theta_i(t_k, t_{k+1}, \ldots, t_{2k}) * u_{i_1} \cdots u_{i_k}\right) \cdots \\ \left(\sum \theta_i(t_{k(d-1)}, t_{k(d-1)+1}, \ldots, t_{kd}) * u_{i_1} \cdots u_{i_k}\right). \end{gathered} \tag{6}$$

Algorithm 2.2. Let $\varphi = (f, g)$ be a z-endomorphism of $K\langle x, y, z\rangle$. We make use of the bidegree defined in the proof of Theorem 1.1.

Step 0. If some of the polynomials f, g depends on z only, then φ is not an automorphism.

Step 1. Let u, v be the homogeneous components of highest bidegree of f, g, respectively. If both u, v are of bidegree $(1, 0)$, i.e., linear, then we check whether they are linearly independent. If yes, then φ is a product of a linear automorphism (from $GL_2(K)$) and a translation $(x + p(z), y + r(z))$. If u, v are linearly dependent, then φ is not an automorphism.

Step 2. Let bideg $u > (1, 0)$ and bideg $u \geq$ bideg v. Hence $u \in H_l$, $v \in H_k$ for some k and l. Taking into account (3), we have to check whether $l = kd$ for a positive integer d and to decide whether $u = q(v/\beta, z)$ for some $\beta \in K^*$ and some $q(y, z) \in H_d$. In the notation of Lemma 2.1, we know u in (6) and v in (4) up to the multiplicative constant β. Hence, up to β, we know the polynomials $\theta_i(t_0, t_1, \ldots, t_n)$ in the presentation of v. We compare some of the nonzero polynomial coefficients of $u = \sum \lambda_j(t_0, \ldots, t_{kd}) u_{j_1} \cdots u_{i_{kd}}$ with the corresponding coefficient of $q(v/\beta, z)$. Lemma 2.1 allows to find explicitly, up to the value of β^d, the polynomial $\omega(t_0, t_1, \ldots, \omega_d)$ in (5) using the usual

division of polynomials. If $l = kd$ and $u = q(v/\beta, z)$, then we replace $\varphi = (f, g)$ with $\varphi_1 = (f - q(g/\beta, z), g)$. Then we apply Step 0 to φ_1. If u cannot be presented in the desired form, then φ is not an automorphism.

Step 3. If bideg $v > (1, 0)$ and bideg $u <$ bideg v, we have similar considerations, as in Step 2, replacing $\varphi = (f, g)$ with $\varphi_1 = (f, g - q(f/\alpha, z))$ for suitable $q(y, z)$. Then we apply Step 0 to φ_1. If v cannot be presented in this form, then φ is not an automorphism.

Corollary 2.3. *Let $h(t, z) \in K\langle t, z\rangle$ and let $\deg_u h(u, z) > 0$. Then*

$$\sigma_h = (x + zh(xz - zy, z), y + h(xz - zy, z)z, z)$$

is a z-wild automorphism of $K\langle x, y, z\rangle$.

Proof. It is easy to see that σ_h is a z-automorphism of $K\langle x, y, z\rangle$ with inverse σ_{-h}. We apply Algorithm 2.2. Let w be the homogeneous component of highest bidegree of $h(xz - zy, z)$. Clearly, w has the form $w = \overline{h}(xz - zy, z) = q(xz - zy, z)$ for some bihomogeneous polynomial $q(t, z) \in K\langle t, z\rangle$. The leading components of the coordinates of σ_h are $zq(xz - zy, z)$ and $q(xz - zy, z)z$, and are of the same bidegree. If σ_h is a z-tame automorphism, then we can reduce the bidegree using a linear transformation, which is impossible because $zq(xz - zy, z)$ and $q(xz - zy, z)z$ are linearly independent. □

The algorithm in Theorem 6.8.5 in [C] which recognizes the automorphisms of $K[x, y]$ can be easily modified to recognize the coordinates of $K[x, y]$. Such an algorithm is explicitly stated in [SY3], where Shpilrain and Yu established an algorithm which gives a canonical form, up to automorphic equivalence, of a class of polynomials in $K[x, y]$. (The automorphic equivalence problem for $K[x, y]$ asks how to decide whether, for two given polynomials $p, q \in K[x, y]$, there exists an automorphism φ such that $q = \varphi(p)$. It was solved over $\mathbb{C}$ by Wightwick [Wi] and, over an arbitrary algebraically closed constructive field K, by Makar-Limanov, Shpilrain, and Yu [MLSY].) When char $K = 0$, Shpilrain and Yu [SY1] gave a very simple algorithm which decides whether a polynomial $f(x, y) \in K[x, y]$ is a coordinate. Their approach is based on an idea of Wright [Wr] and the Euclidean division algorithm applied for the partial derivatives of a polynomial in $K[x, y]$. Using the isomorphism of Aut $K[x, y]$ and Aut $K\langle x, y\rangle$ and reducing the considerations to the case of $K[x, y]$, Shpilrain and Yu [SY2] found the first algorithm which recognizes the coordinates of $K\langle x, y\rangle$. Now we want to modify

Algorithm 2.2 to decide whether a polynomial $f(x,y,z)$ is a z-tame coordinate of $K\langle x,y,z\rangle$.

Note, that if $\varphi=(f,g)$ and $\varphi'=(f,g')$ are two z-automorphisms of $K\langle x,y,z\rangle$ with the same first coordinate, then $\varphi^{-1}\varphi'$ fixes x. Hence $\varphi^{-1}\varphi'=(x,g'')$ and, obligatorily, $g''=\beta y+r(x,z)$. In this way, if we know one z-coordinate mate g of f, then we are able to find all other z-coordinate mates. These arguments and Corollary 2.3 give immediately:

Corollary 2.4. *Let $h(t,z)\in K\langle t,z\rangle$ and let $\deg_u h(u,z)>0$. Then $f(x,y,z)=x+zh(xz-zy,z)$ is a z-wild coordinate of $K\langle x,y,z\rangle$.*

Theorem 2.5. *There is an algorithm which decides whether a polynomial $f(x,y,z)\in K\langle x,y,z\rangle$ is a z-tame coordinate.*

Proof. We start with the analysis of the behavior of the first coordinate f of φ in (2). Let h be the first coordinate of $\psi=\rho_{n-1}\tau\cdots\tau\rho_1\tau\rho_0$ and let, as in (2), $\rho_n=(x+p_n(y,z),y)$ and $p_n(0,z)=0$. Then

$$f(x,y,z)=\rho_n\tau(h(x,y,z))=h(y,x+p_n(y,z),z). \tag{7}$$

In order to make the inductive step, we have to recover the polynomials $h(x,y,z)$ and $p_n(y,z)$ or, at least their leading components with respect to a suitable grading.

For a pair of positive integers (a,b), we define the (a,b)-bidegree of a monomial $w\in K\langle x,y,z\rangle$ by

$$\text{bideg}_{(a,b)}w=(a\deg_x w+b\deg_y w,\deg_z w)$$

and order the bidegrees in the lexicographic order, as in Algorithm 2.2. For a nonzero polynomial $f\in K\langle x,y,z\rangle$ we denote by $|f|_{(a,b)}$ the homogeneous component of maximal (a,b)-bidegree. We write $\varphi=(f,g)\in \text{TAut}_z K\langle x,y,z\rangle$ in the form (2). Let us assume again that bideg $p_i(y)>(1,0)$ for all $i=0,1,\ldots,n$, and let h be the first coordinate of $\psi=\rho_{n-1}\tau\cdots\tau\rho_1\tau\rho_0$. Then the highest bihomogeneous component of h is

$$\overline{h}(y,z)=q_0(q_1(\ldots(q_{n-1}(y,z),z)\ldots),z).$$

The homogeneous component of maximal $(d_n,1)$-bidegree of $x+p_n(y,z)$ is $|x+q_n(y,z)|_{(d_n,1)}=x+\xi_n y^{d_n}$ if $\deg_z q_n(y,z)=0$ and $|x+q_n(y,z)|_{(d_n,1)}=q_n(y,z)$ if $\deg_z q_n(y,z)>0$. Direct calculations give

$$|f|_{(d_n,1)}=|\rho_n\tau(\overline{h})|_{(d_n,1)}=|\overline{h}(x+q_n(y,z))|_{(d_n,1)}.$$

If $f'(x,z)$ and $f''(y,z)$ are the components of $f(x,y,z)$ which do not depend on y and x, respectively, we can recover the degree d_n of $p_n(y,z)$ as the quotient $d_n = \deg_x f'/\deg_y f''$. Now the problem is to recover $q_n(y,z)$ and $\overline{h}(y,z)$. Since $\overline{h}(y,z)$ does not depend on x, we have that

$$\overline{h}(y,z) = \overline{h(x,y,z)} = \overline{h(0,y,z)}.$$

From the equality (7) and the condition $p_n(0,z) = 0$ we obtain that

$$f(x,0,z) = h(0, x + p_n(0,z), z) = h(0,x,z).$$

Hence $h(0,y,z) = f(y,0,z)$ and we are able to find $\overline{h}(y,z)$. We write $\overline{h}$ and $\overline{q_n}$ in the form

$$\overline{h}(y,z) = \theta(t_0,t_1,\ldots,t_k) * y^k, \quad q_n(y,z) = \omega(t_0,t_1,\ldots,t_d) * y^d,$$

where $\theta(t_0,t_1,\ldots,t_k) \in K[t_0,t_1,\ldots,t_k]$ is known explicitly and $\omega(t_0,t_1,\ldots,t_d) \in K[t_0,t_1,\ldots,t_d]$. Similarly, the part of the component of maximal bidegree of $f(x,y,z)$ which does not depend on x has the form

$$\overline{f''}(y,z) = \zeta(t_0,t_1,\ldots,t_{kd}) * y^{kd}, \quad \zeta(t_0,t_1,\ldots,t_{kd}) \in K[t_0,t_1,\ldots,t_{kd}].$$

Since $\overline{h}(q_n(y,z),z) = \overline{f''}(y,z)$, by Lemma 2.1 we obtain

$$\zeta(t_0,t_1,\ldots,t_{kd}) = \theta(t_0,t_d,t_{2d},\ldots,t_{kd})\omega(t_0,t_1,\ldots,t_d)$$
$$\omega(t_d,t_{d+1},\ldots,t_{2d})\cdots\omega(t_{(k-1)d},t_{(k-1)d+1},\ldots,t_{kd}).$$

Here we know ζ and θ and want to determine ω. Let

$$\zeta'(t_0,t_1,\ldots,t_{kd}) = \zeta(t_0,t_1,\ldots,t_{kd})/\theta(t_0,t_d,t_{2d},\ldots,t_{kd})$$
$$= \omega(t_0,t_1,\ldots,t_d)\omega(t_d,t_{d+1},\ldots,t_{2d})\cdots\omega(t_{(k-1)d},t_{(k-1)d+1},\ldots,t_{kd}).$$

The greatest common divisor of the polynomials $\zeta'(t_0,t_1,\ldots,t_{kd})$ and $\zeta'(t_{(k-1)d},t_{(k-1)d+1},\ldots,t_{(2k-1)d})$ in $K[t_0,t_1,\ldots,t_{(2k-1)d}]$ is equal, up to a multiplicative constant β, to $\omega(t_{(k-1)d},t_{(k-1)d+1},\ldots,t_{kd})$. Hence the knowledge of ζ' allows to determine $\beta\omega(t_0,t_1,\ldots,t_k)$ as well as the value of β^d. This means that we know also all the possible values of β and the polynomial $q_n(y,z)$. Now we apply on $f(x,y,z)$ the z-automorphism $\sigma = (x - q_n(y,z), y)$. Since $f(x,y,z) - \overline{h}(x + q_n(y,z),z)$ is lower in the $(d_n,1)$-biordering than $f(x,y,z)$ itself, we may replace f with $\sigma(f)$ and to make the next step. The considerations are almost the same when some of the automorphisms ρ_0 and ρ_n is affine. For example, if $f = \varphi(x)$ and $\rho_n = (x + \gamma y, y)$, $\gamma \in K$, in (2), then the leading bihomogeneous component of $h = \tau\rho_n^{-1}(f)$ does not depend on y, and we can do the next step. If f is a z-tame coordinate, then the above process will stop when we reduce f to a polynomial in the form $\alpha x + p(y,z)$. If f is not

a z-tame coordinate, then the process will also stop by different reason. In some step we shall reduce $f(x,y,z)$ to a polynomial $f_1(x,y,z)$. It may turn out that the degree $d = \deg_x f_1(x,0,z)/\deg_y f_1(0,y,z)$ is not integer. Or, the commutative polynomials θ and ω corresponding to f_1 do not exist. $\square$

The following corollary is stronger than Corollary 2.4.

Corollary 2.6. *Let $h(t,z) \in K\langle t,z\rangle$ and let $\deg_u h(u,z) > 0$. Then $f(x,y,z) = x + h(xz - zy, z)$ is not a z-tame coordinate of $K\langle x,y,z\rangle$.*

Proof. We apply the algorithm in the proof of Theorem 2.5. Let $f(x,y,z)$ be a z-tame coordinate and let $h'(x,z) = h(xz,z)$ and $h''(y,z) = h(-zy,z)$ be the polynomials obtained from $h(xz-zy,z)$ replacing, respectively, y and x by 0. Clearly, $\mathrm{bideg}_x h' = \mathrm{bideg}_y h''$. Hence, as in the proof of Theorem 2.5 we can replace $f(x,y,z)$ with $\sigma(f)$, where $\sigma = (x - \alpha y, y)$, for a suitable $\alpha \in K^*$, and the leading bihomogeneous component of $\sigma(f)$ in the $(1,1)$-ordering does not depend on y. But this brings to a contradiction. If $h_1(t,z) \in K\langle t,z\rangle$ is homogeneous with respect to t, and

$$h_1((x - \gamma y)z - zy, z) = h_2(x,z)$$

for some $h_2(x,z)$, then, replacing x with 0, we obtain $h_1(-(\gamma yz + zy), z) = 0$, which is impossible. $\square$

Remark 2.7. In Corollary 2.6, we cannot guarantee that the polynomial $f(x,y,z) = x + h(xz - zy, z)$ is a z-coordinate at all. For example, let $f(x,y,z) = x + (xz - zy)$ be a z-coordinate with a coordinate mate $g(x,y,z)$. If $g_1(x,y,z)$ is the linear in x,y component of g, then $\varphi_1 = (f, g_1)$ is also a z-automorphism. Then, for suitable polynomials $c, d \in K[z_1, z_2]$, the matrix

$$J_z(\varphi_1) = \begin{pmatrix} 1 + z_2 & c(z_1,z_2) \\ -z_1 & d(z_1,z_2) \end{pmatrix}$$

is invertible. If we replace z_1 with 0 in its determinant $\det(J_z) = (1 + z_2)d(z_1,z_2) - z_1 c(z_1,z_2)$ we obtain that $(1 + z_2)d_2(0,z_2) \in K^*$ which is impossible.

3. Endomorphisms which are not automorphisms

In this section we shall establish a z-analogue of the following proposition which is the main step of the proof of the theorem of Czerniakiewicz [Cz] and Makar-Limanov [ML1, ML2] for the tameness of the automorphisms of $K\langle x, y\rangle$.

Proposition 3.1. *Let $\varphi = (x+u, y+v)$ be an endomorphism of $K\langle x, y\rangle$, where u, v are in the commutator ideal of $K\langle x, y\rangle$ and at least one of them is different from 0. Then φ is not an automorphism of $K\langle x, y\rangle$.*

An essential moment in its proof, see the book by Cohn [C], is the following lemma.

Lemma 3.2. *If $f, g \in K\langle x, y\rangle$ are two bihomogeneous polynomials, then they either generate a free subalgebra of $K\langle x, y\rangle$ or, up to multiplicative constants, both are powers of the same bihomogeneous element of $K\langle x, y\rangle$.*

We shall prove a weaker version of the lemma for $K\langle x, y, z\rangle$ which will be sufficient for our purposes.

Lemma 3.3. *Let $(0,0) \neq (a,b) \in \mathbb{Z}^2$ and let $f_1, f_2 \in K\langle x, y, z\rangle$ be bihomogeneous with respect to the (a,b)-degree of $K\langle x, y, z\rangle$, i.e., $a\deg_x w + b\deg_y w$ is the same for all monomials of f_1, and similarly for f_2. If f_1 and f_2 are algebraically dependent, then both $\deg_{(a,b)} f_1$ and $\deg_{(a,b)} f_2$ are either nonnegative or nonpositive.*

Proof. Let $v(f_1, f_2, z) = 0$ for some nonzero polynomial $v(u_1, u_2, z) \in K\langle u_1, u_2, z\rangle$. We may assume that both f_1, f_2 depend not on z only. We fix a term-ordering on $K\langle x, y, z\rangle$. Let $\tilde{f}_1$ and $\tilde{f}_2$ be the leading monomials of f_1 and f_2, respectively. For each monomial $z^{k_0}u_{i_1}z^{k_1}\cdots z^{k_{s-1}}u_{i_s}z^{k_s} \in K\langle u_1, u_2, z\rangle$ the leading monomial of $z^{k_0}f_{i_1}z^{k_1}\cdots z^{k_{s-1}}f_{i_s}z^{k_s} \in K\langle x, y, z\rangle$ is $z^{k_0}\tilde{f}_{i_1}z^{k_1}\cdots z^{k_{s-1}}\tilde{f}_{i_s}z^{k_s}$. Hence, the algebraic dependence of f_1 and f_2 implies that two different monomials $z^{k_0}\tilde{f}_{i_1}z^{k_1}\cdots z^{k_{s-1}}\tilde{f}_{i_s}z^{k_s}$ and $z^{l_0}\tilde{f}_{j_1}z^{l_1}\cdots z^{l_{t-1}}\tilde{f}_{j_t}z^{j_t}$ are equal. We write $\tilde{f}_1 = z^{p_1}g_1z^{q_1}$ and $\tilde{f}_2 = z^{p_2}g_2z^{q_2}$, where g_1, g_2 do not start and do not end with z. After some cancelation in the equation

$$z^{k_0}\tilde{f}_{i_1}z^{k_1}\cdots z^{k_{s-1}}\tilde{f}_{i_s}z^{k_s} = z^{l_0}\tilde{f}_{j_1}z^{l_1}\cdots z^{l_{t-1}}\tilde{f}_{j_t}z^{j_t}$$

we obtain a relation of the form

$$g_{a_1}z^{m_1}\cdots z^{m_{k-1}}g_{a_k}z^{m_k} = g_{b_1}z^{n_1}\cdots z^{n_{l-1}}g_{b_l}z^{n_l}, \tag{8}$$

with different g_{a_1} and g_{b_1}. Hence, if $\deg g_1 \geq \deg g_2$, then $g_1 = g_2g_3$ for some monomial g_3 (and $g_2 = g_1g_3$ if $\deg g_1 < \deg g_2$). Again, g_2 and g_3 satisfy a relation of the form (8). Since $\deg g_1 \geq \deg g_2 > 0$, we obtain $\deg g_1 + \deg g_2 > \deg g_1 = \deg g_2 + \deg g_3$. Applying inductive arguments, we derive that both $\deg_{(a,b)} g_2$ and $\deg_{(a,b)} g_3$ are either nonnegative or nonpositive, and the same holds for f_1 and f_2 because $g_1 = g_2g_3$, $\deg_{(a,b)} g_1 = \deg_{(a,b)} g_2 + \deg_{(a,b)} g_3$, and $\deg_{(a,b)} f_i = \deg_{(a,b)} g_i$, $i = 1, 2$. □

The condition that $u(x, y)$ and $v(x, y)$ belong to the commutator ideal of $K\langle x, y\rangle$, as in Proposition 3.1, immediately implies that all monomials of u and v depend on both x and y, as required in the following theorem.

Theorem 3.4. *The z-endomorphisms of the form*

$$\varphi = (x + u(x, y, z), y + v(x, y, z)),$$

where $(u, v) \neq (0, 0)$ and all monomials of u and v depend on both x and y, are not automorphisms of $K\langle x, y, z\rangle$.

Proof. The key moment in the proof of Proposition 3.1 is the following. If $\varphi = (x + u, y + v)$ is an endomorphism of $K\langle x, y\rangle$, where u, v are in the commutator ideal of $K\langle x, y\rangle$ and at least one of them is different from 0, then there exist two integers a and b such that $(a, b) \neq (0, 0)$ and $a \leq 0 \leq b$ with the property that $\deg_{(a,b)}(x + u) = \deg_{(a,b)} x = a$ and $\deg_{(a,b)}(y + v) = \deg_{(a,b)} y = b$. Ordering in a suitable way the (a, b)-bidegrees, one concludes that the (a, b)-degrees of the leading bihomogeneous components of $x + u$ and $y + v$ are with different signs. Then Lemma 3.2 shows that these leading components are algebraically independent and bidegree arguments as in the proof of Proposition 3.1 give that φ cannot be an automorphism. We repeat verbatim these arguments, working with the same (a, b)-(bi)degree and bidegree ordering Proposition 3.1, without counting the degree of z. In the final step, we use Lemma 3.3 instead of Lemma 3.2. □

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